Fiber-Base duality, Global Symmetry Enhancement and Gopakumar-Vafa invariant

Futoshi Yagi (Technion)

Based on
arXiv: 1411.2450: V. Mitev, E.Pomoni, M.Taki, FY
Work in progress: H.Hayashi, S-S.Kim, K.Lee, M.Taki, FY

5D $\mathcal{N}=1$ SUSY SU(2) gauge theory with N_f flavor

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5D UV fixed point exists for $N_f \leqq 7$ ’96 Seiberg

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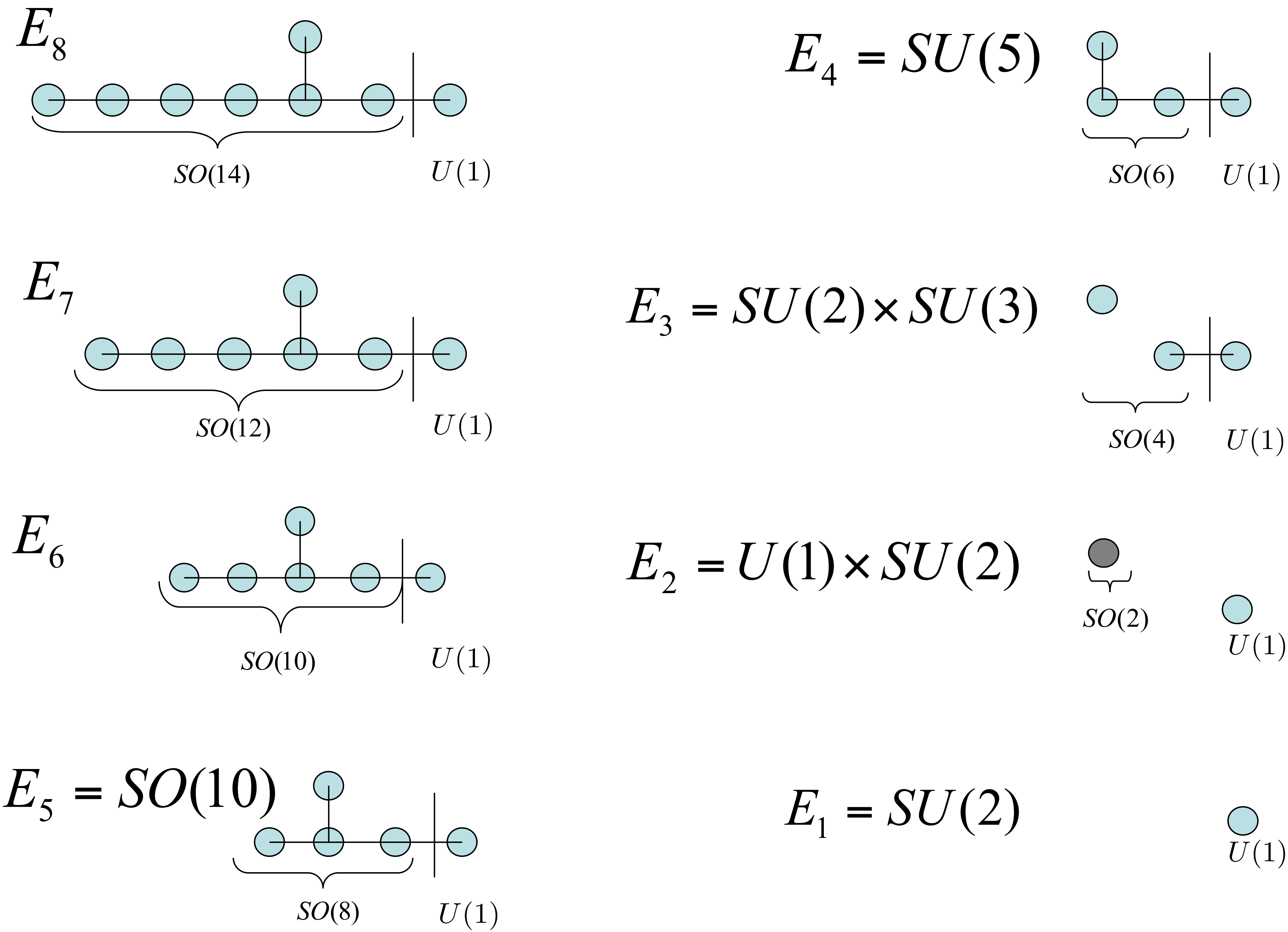
Global symmetry enhancement at UV fixed point

$$SO(2N_f) \times U(1) \subset E_{N_f+1}$$

N_f flavors

Instanton particle



E_8
$SO(14)$
$U(1)$
$E_4 = SU(5)$
$SO(6)$
$U(1)$
E_7
$SO(12)$
$U(1)$
$E_3 = SU(2) \times SU(3)$
$SO(4)$
$U(1)$
E_6
$SO(10)$
$U(1)$
$E_2 = U(1) \times SU(2)$
$SO(2)$
$U(1)$
$E_5 = SO(10)$
$SO(8)$
$U(1)$
$E_1 = SU(2)$
$U(1)$

Can we see global symmetry enhancement from brane web?

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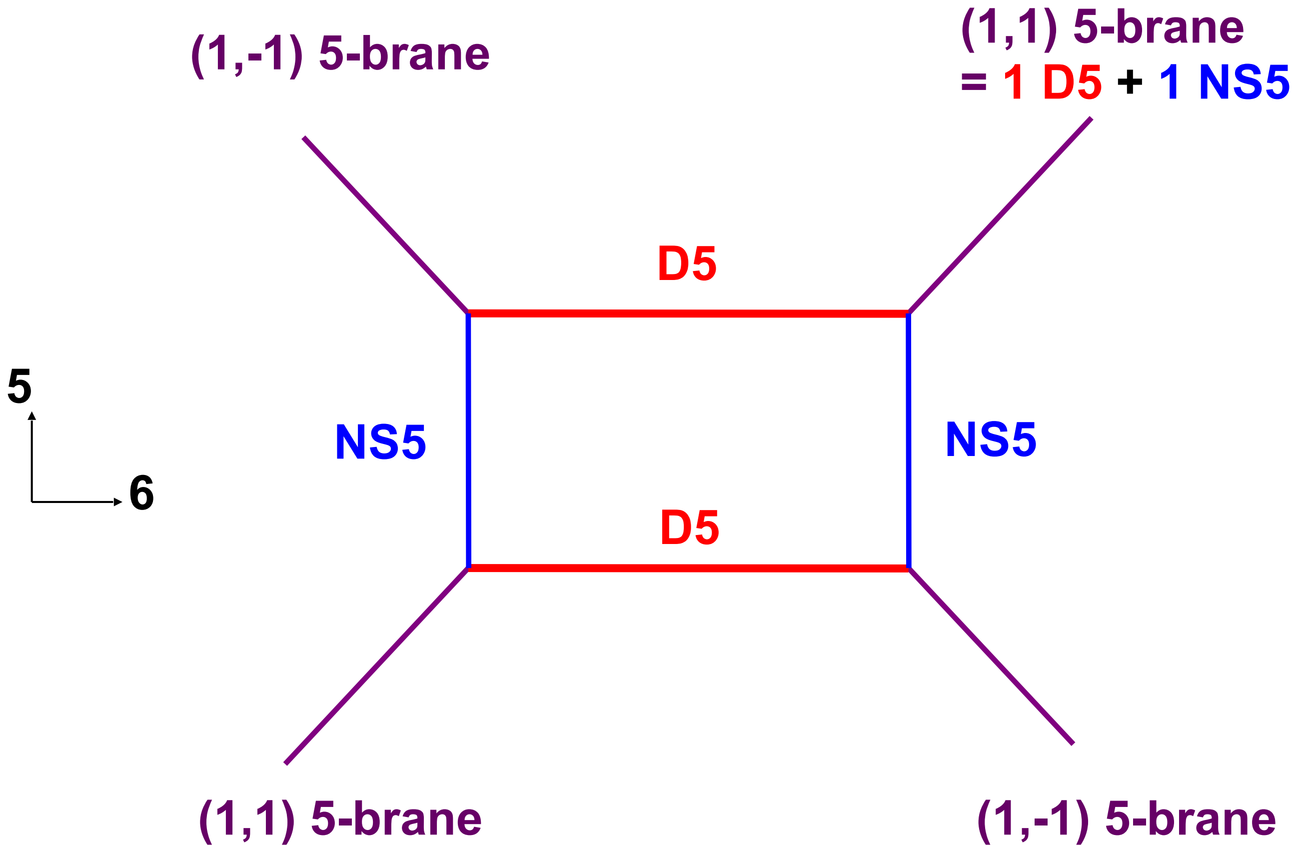
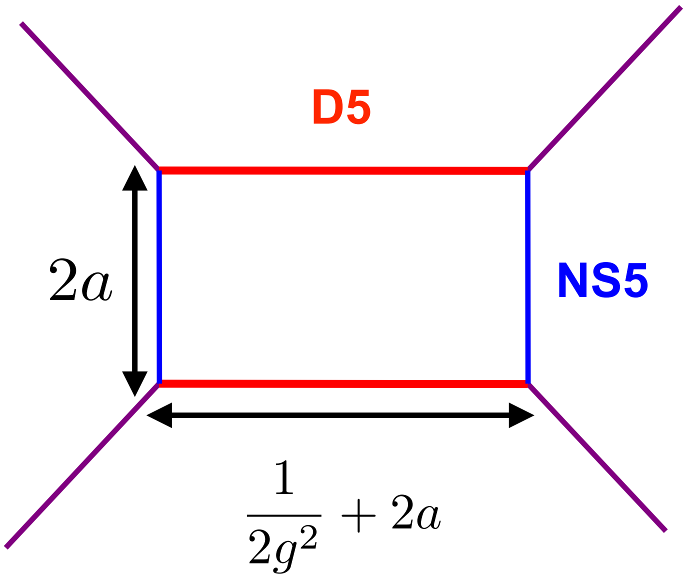
→ S-duality
(Fiber-base duality in CY language)

Brane setup for pure SU(2) SYM



NS5	0 1 2 3 4 5
D5	0 1 2 3 4 6

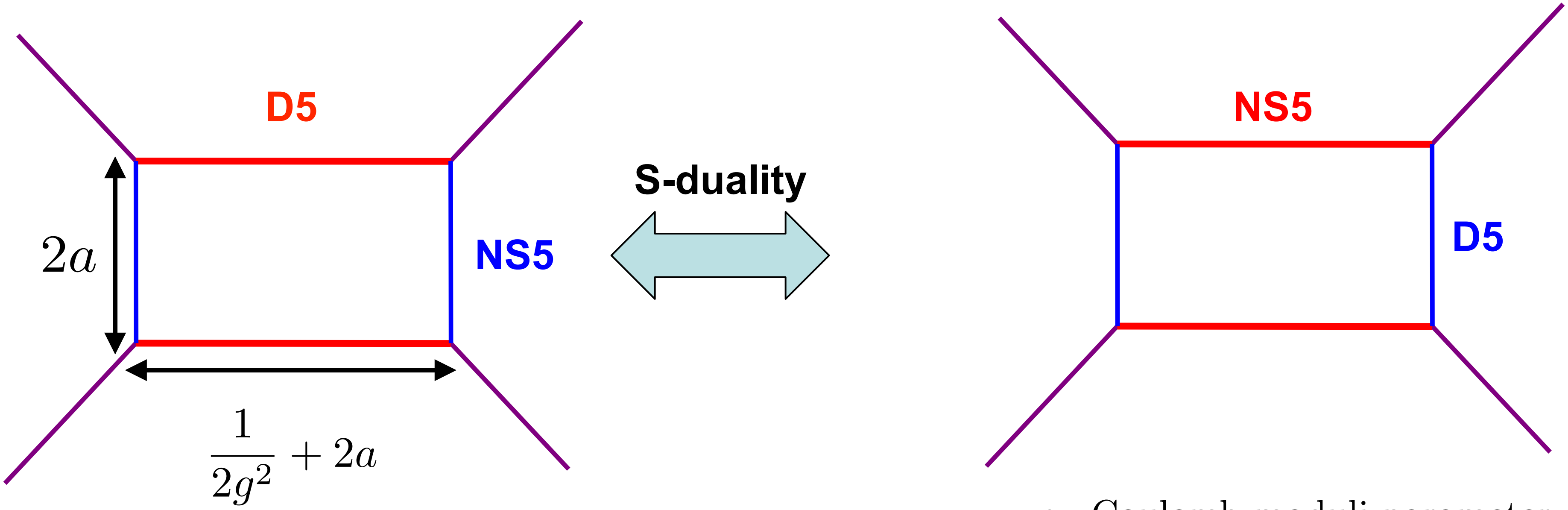
pure SU(2) SYM



a : Coulomb moduli parameter

g : (Bare) gauge coupling

S-duality for pure SU(2) SYM



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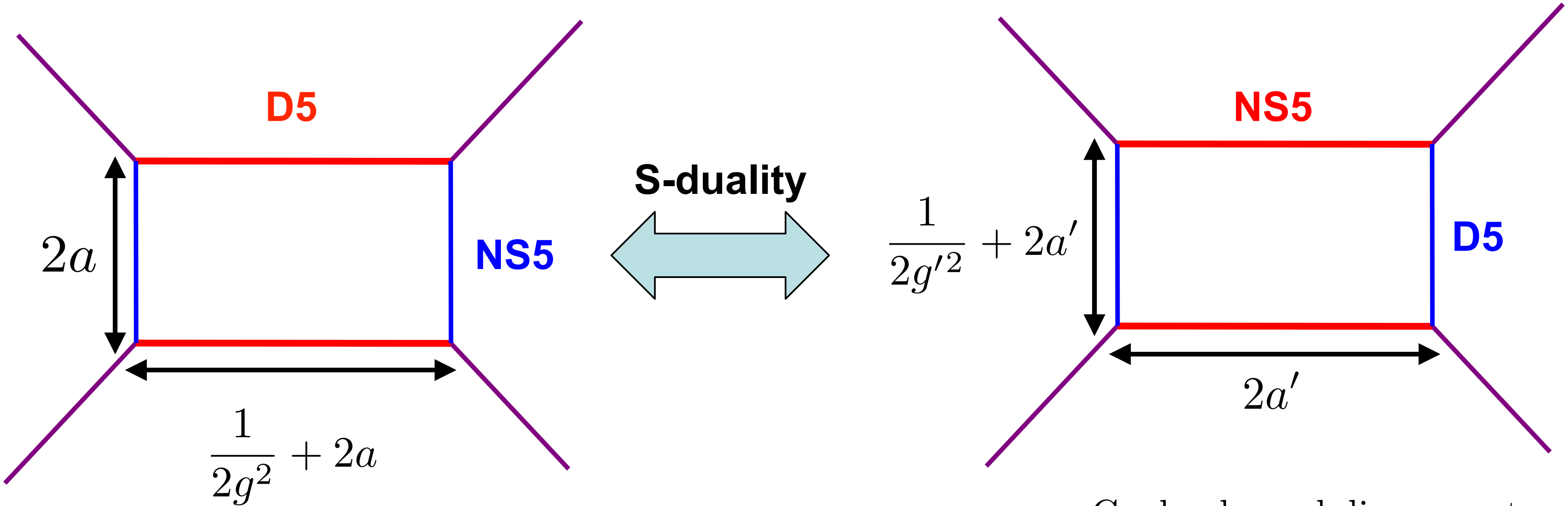

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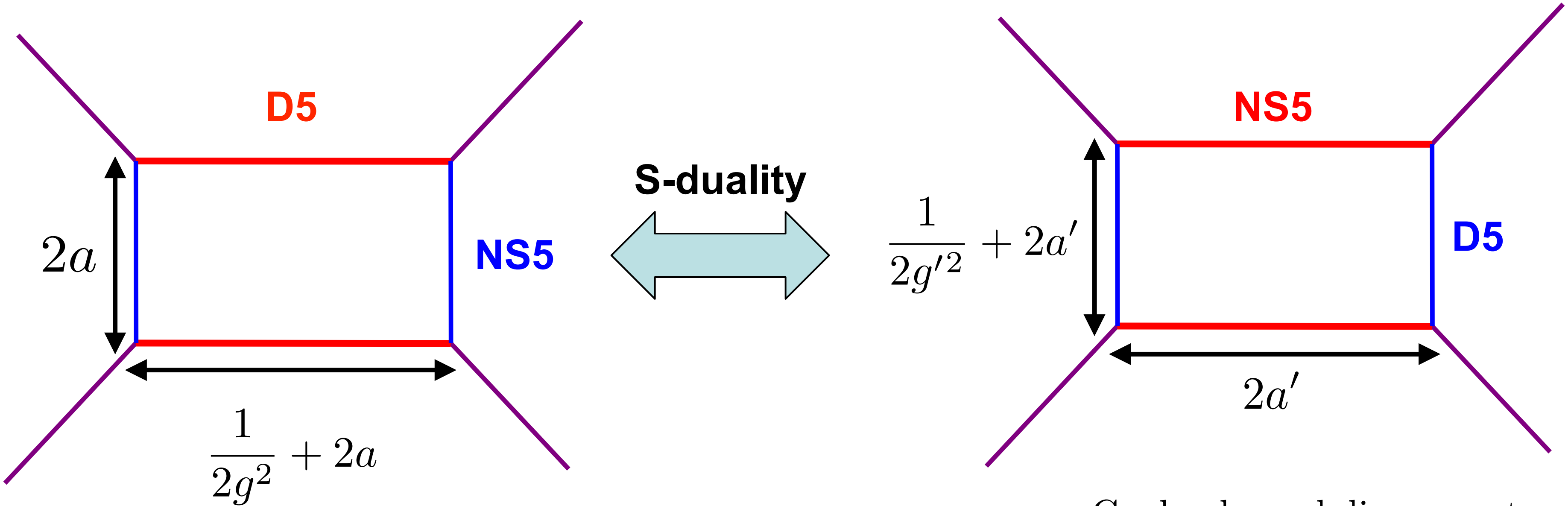

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$$\frac{1}{g'^2} = -\frac{1}{g^2}$$

$$a' = a + \frac{1}{4g^2}$$

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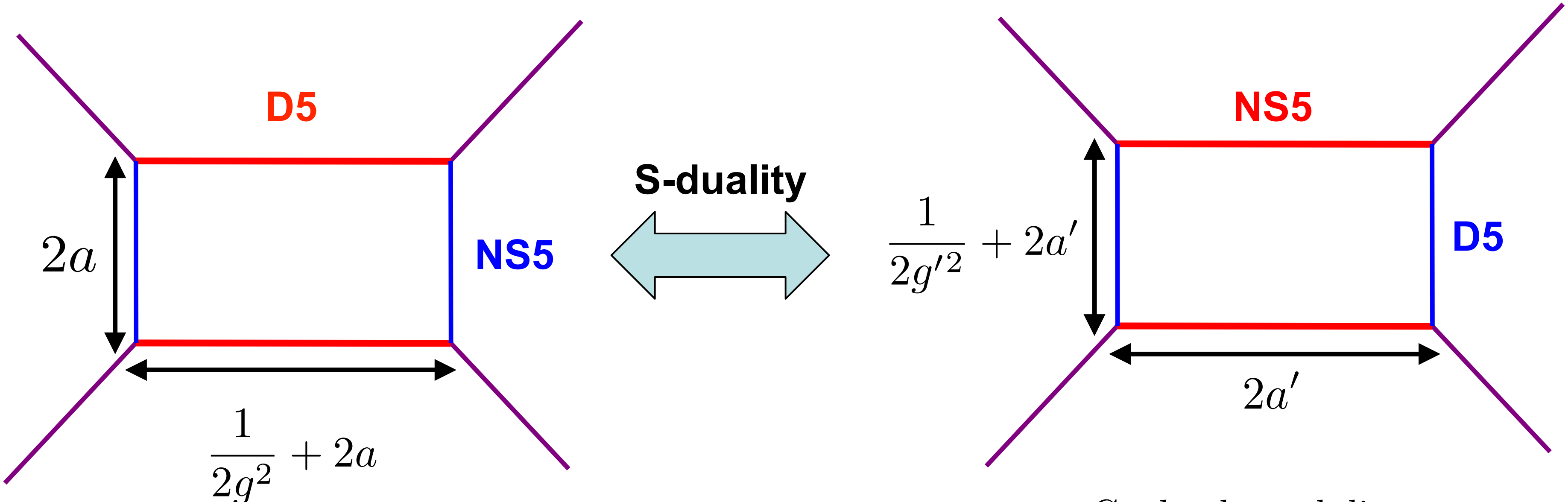

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Weyl Symmetry for $E_1 = SU(2)$

'97 Aharony,Hanany,Kol

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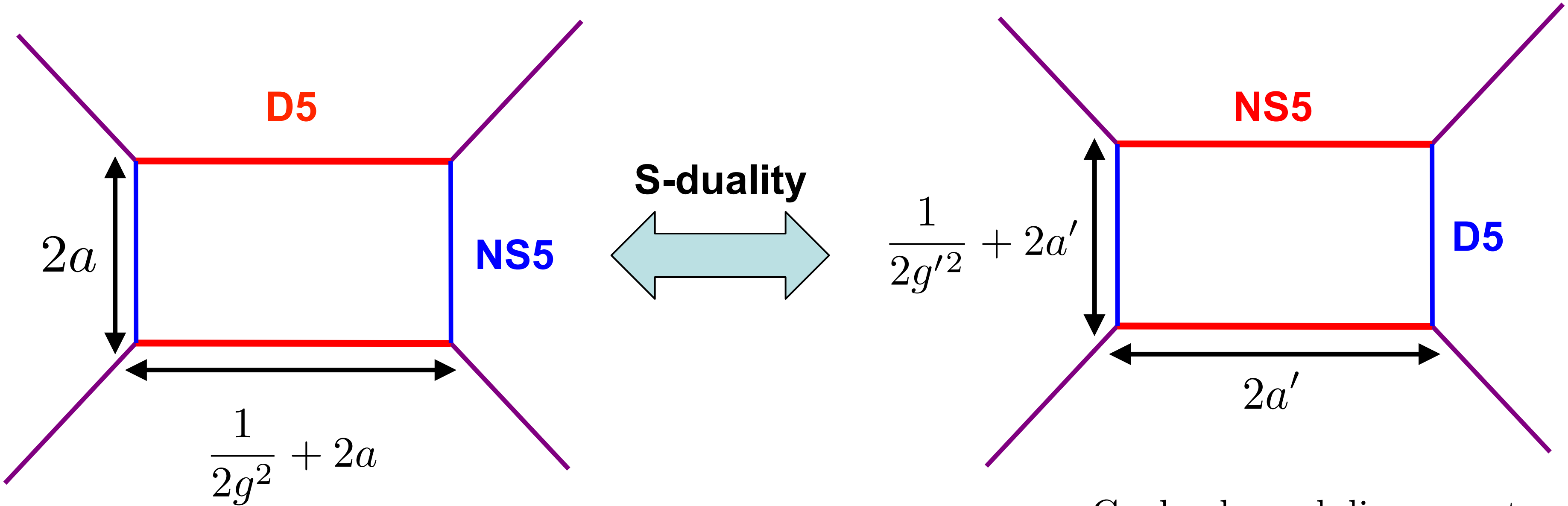

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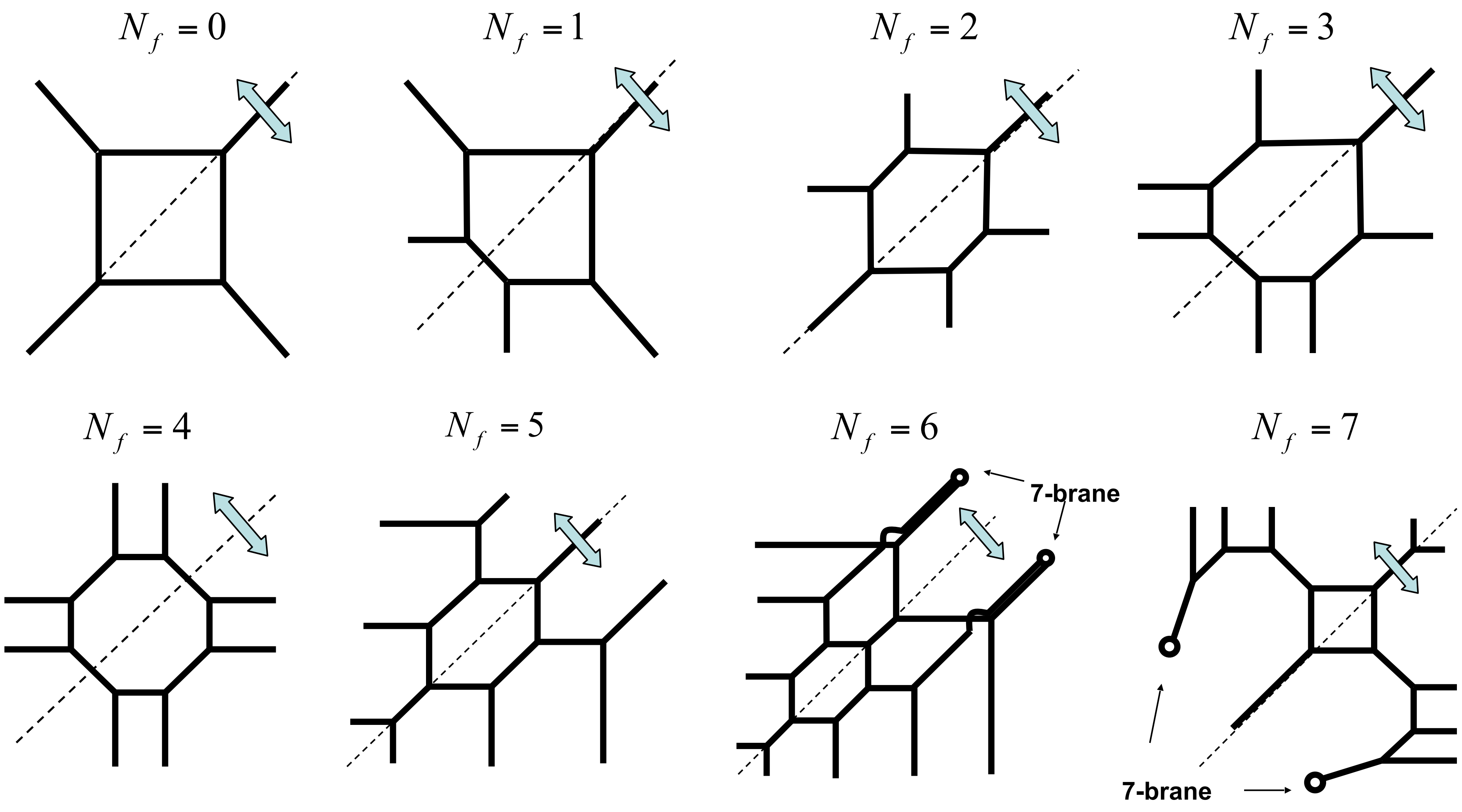
$$a' = a + \frac{1}{4g^2}$$

Coulomb moduli parameter is also transformed!

Generalization to higher flavor



'09 Benini-Benvenuti-Tachikawa

(Weyl symmetry of) $\mathbf{SO(2N_f) \times U(1)}$	+	Transformation induced from **S-duality**

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Transformation induced from
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Flavors
(Masses)

Instanton particle
(Gauge coupling)

Flavors $\leftrightarrow$ Instanton particle
(Masses $\leftrightarrow$ Gauge coupling)

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$\Rightarrow$

(Weyl symmetry of)
Enhanced $\mathbf{E_{N_f+1}}$ symmetry

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Flavors $\leftrightarrow$ Instanton particle
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$\Rightarrow$

(Weyl symmetry of)
Enhanced $\mathbf{E_{N_f+1}}$ symmetry

Again, Coulomb moduli parameter is also transformed!

Can we write Nekrasov partition function in manifestly E_{Nf+1} invariant way ?

Original Nekrasov partition function does not look manifestly E_{Nf+1} invariant because…

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1. Use invariant variable $\tilde{A} = q^{\frac{2}{8-N_f}} e^{-\beta a}$ instead of $e^{-\beta a}$

$$\left(\frac{1}{g^2} \to -\frac{1}{g^2}, \quad a \to a + \frac{1}{4g^2}, \quad \tilde{A} = q^{\frac{1}{4}} e^{\beta a} = e^{\beta (a + \frac{1}{8g^2})} \quad (N_f = 0) \right)$$

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2. Expand in terms of $\tilde{A}$

E_{N_f+1} invariant Nekrasov partition function

$$Z(a,g,m_i;\epsilon_1,\epsilon_2)=Z_{pert}(a,m_i;\epsilon_1,\epsilon_2)\sum_{k=0}^{\infty}Z_k(a,m_i;\epsilon_1,\epsilon_2)q^k$$

Original form

'12 H-C Kim, S-S.Kim, K.Lee
'14 C.Hwang, J.Kim, S.Kim, J.Park

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$$= \sum_{n=0}^{\infty} \underbrace{\tilde{Z}_n(g,m_i;\epsilon_1,\epsilon_2)}_{E_{Nf+1}\text{ invariant}} \tilde{A}^k$$

New form

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New form

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$$= \exp\left[\sum_{k=0}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\tilde{F}_n(kg,km_i;k\epsilon_1,k\epsilon_2)\tilde{A}^{nk}\right]$$

$$\equiv \mathrm{PE}\left[\sum_{n=1}^{\infty}\underbrace{\tilde{F}_n(g,m_i;\epsilon_1,\epsilon_2)}_{E_{N_f+1}\text{ invariant}}\tilde{A}^n\right]$$

Nekrasov partition function for pure $SU(2)$

$$Z = \mathrm{PE}\left[\frac{\mathfrak{q}+\mathfrak{t}}{(1-\mathfrak{q})(1-\mathfrak{t})}\chi_2^{E_1}\tilde{A}^2 + \mathcal{O}(\tilde{A}^4)\right]$$

Character of $E_1 = SU(2):\quad \chi_2^{E_1} = q^{\frac{1}{2}} + q^{-\frac{1}{2}},$

$$\tilde{A} = q^{\frac{1}{4}}e^{\beta a}, \quad \mathfrak{q} = e^{-\beta\epsilon_1}, \quad \mathfrak{t} = e^{\beta\epsilon_2}$$

Manifestly E_1 invariant!!

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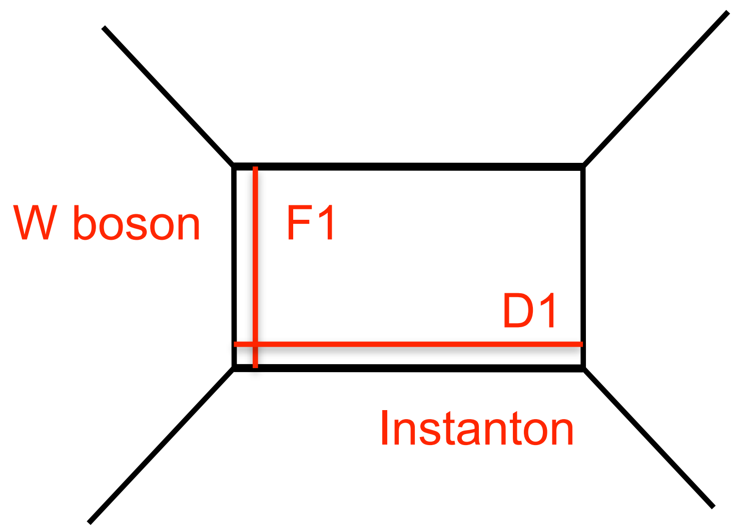

Nekrasov partition function for N_f=1

$$Z = \mathrm{PE}\Bigg[-\frac{\mathfrak{q}^{\frac{1}{2}}\mathfrak{t}^{\frac{1}{2}}}{(1-\mathfrak{q})(1-\mathfrak{t})}\left(\chi_2^{SU(2)}(u_1){u_2}^{-\frac{3}{7}} + {u_2}^{\frac{4}{7}}\right)\tilde{A}$$

$$+\frac{\mathfrak{q}+\mathfrak{t}}{(1-\mathfrak{q})(1-\mathfrak{t})}\chi_2^{SU(2)}(u_1)u_2^{\frac{1}{7}}\tilde{A}^2 + \mathcal{O}(\tilde{A}^3)\Bigg]$$

$$E_2 = SU(2)\times U(1)$$

$SU(2):\quad u_1 = q^{\frac{1}{2}}e^{-\frac{1}{4}\beta m}\qquad \chi_2(u_1) = u_1 + {u_1}^{-1}$

$U(1):\quad u_2 = q^{-\frac{1}{2}}e^{-\frac{7}{4}\beta m}\qquad \tilde{A} = q^{\frac{2}{7}}e^{-\beta a}$

Nekrasov partition function for $N_f=1$

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$$+\frac{\mathfrak{q}+\mathfrak{t}}{(1-\mathfrak{q})(1-\mathfrak{t})}\underline{\chi_2^{SU(2)}(u_1)u_2^{\frac{1}{7}}}\,\tilde{A}^2 + \mathcal{O}(\tilde{A}^3)\Bigg]$$

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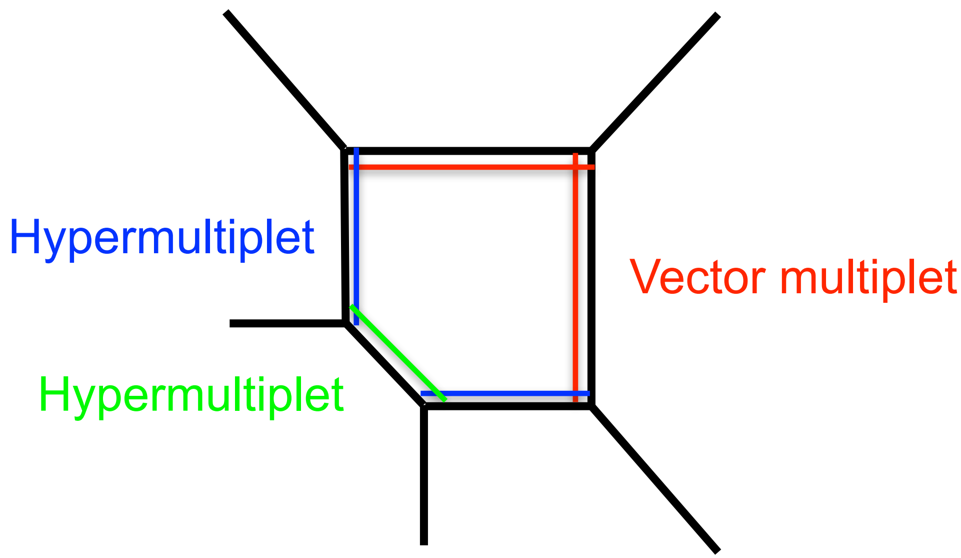
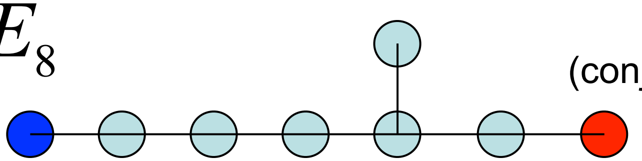
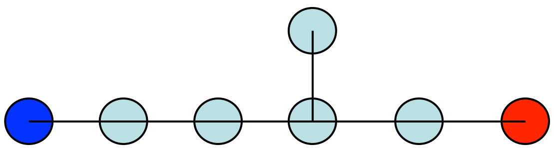


The vector multiplet and the hypermultiplet are included in the fundamental representation of E_{Nf+1} corresponding to the following nodes

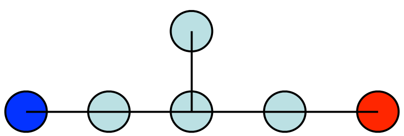
E_8



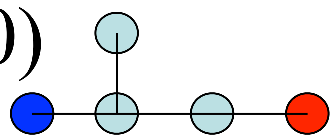
E_7



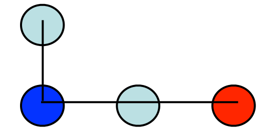
E_6



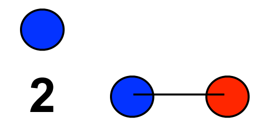
$E_5 = SO(10)$



$E_4 = SU(5)$

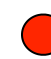


$E_3 = SU(2) \times SU(3)$



$E_2 = U(1) \times SU(2)$

$E_1 = SU(2)$



Gopakumar-Vafa's expansion

$$Z = \mathrm{PE}\left[\sum_{C\in H_2(X,\mathbb{Z})}\sum_{j_L,j_R}\sum_{k_L=-j_L}^{j_L}\sum_{k_R=-j_R}^{j_R}\frac{M_C^{(j_L,j_R)}\mathfrak{t}^{k_L+k_R}\mathfrak{q}^{k_L-k_R}}{(\mathfrak{t}-\mathfrak{t}^{-1})(\mathfrak{q}-\mathfrak{q}^{-1})}Q^C\right]$$

X : Calabi-Yau manifold

$Q^C = e^{-\int_C \omega}$, ω : Kähler form

$(Q^C = e^{-2\beta a}, e^{-\beta(a-m)}, q^k e^{-2\beta a}, \cdots)$

$M_C^{(j_L,j_R)}$: Refined Gopakumar-Vafa invariant

Gopakumar-Vafa '98
Iqbal, Kozcaz, Vafa '07

Nekrasov partition function $\longleftrightarrow$ **Set of integers** $M_C^{(j_L,j_R)}$

Non-Negative integer

(After the convention change $\tilde{A} \to -\tilde{A}$)

Consistent with the result from topological B-model

[Huang, Klemm, Poretschkin '13]

Summary

$$SO(2N_f) \times U(1) + \text{S-duality} = E_{N_f+1}$$

Nekrasov partition function is invariant

Refined Gopakumar-Vafa invariants from Nekrasov partition function agrees with topological B-model computation